Energy asymptotics for strongly damped wave equations
Lublin

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1 D waves

- $$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \qquad t, x \in \mathbb{R} \\ u(x,0) &= f(x), \\ u_t(x,0) &= g(x) \end{aligned}$$

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- Suppose f, g, F, G are real.

Energies

- $$\begin{aligned} K(t) &= \frac{1}{2}\int_{-\infty}^{\infty} u_t(x,t)^2\,dx \\ P(t) &= \frac{1}{2}\int_{-\infty}^{\infty} u_x(x,t)^2\,dx \\ E(t) &= K(t) + P(t) \end{aligned}$$

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- Assume $f, g \in C_c^2(\mathbb{R})$. Energy conservation:

$$\begin{aligned} dE/dt &= \int_{\mathbb{R}} (u_t u_{tt} + u_x u_{xt}) dx \\ &= \int_{\mathbb{R}} (u_t u_{tt} - u_t u_{xx}) dx = 0. \end{aligned}$$

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- $$\begin{aligned} P(t) - K(t) &= 2\int_{\mathbb{R}} F'(x+t)G'(x-t)dx = 0 \\ \text{if } |t| &> R, \text{ since } F'(s) = G'(s) = 0 \text{ for } |s| > R. \end{aligned}$$

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- Thus
$$K(t) = P(t) = \frac{E}{2} = \text{constant for } |t| > R.$$
- This is equipartition of energy.

Equipartion of energy II

- Moreover, for all finite energy solutions,

$$\lim_{t \to \infty} K(t) = \lim_{t \to \infty} P(t) = E/2.$$

Spectral Theorem

Theorem

$A = A^*$ on H iff there is a unitary $U : H \to L^2(\Omega, \Sigma, \mu)$ such that

$$
\begin{aligned}
UAU^{-1} &= M_a \\
a &: \Omega \to \mathbb{R},\ \Sigma - \textit{measurable} \\
M_a g &= ag, \quad D(M_a) = \{u : u, au \in L^2\} \\
\sigma(A) &= \textit{essrange}(a).
\end{aligned}
$$

For any Borel function $h : \sigma(A) \to \mathbb{C}$,

$$
\begin{aligned}
h(A) &= UM_{h(a)}U^{-1}, \\
h(A) &= h(A)^* \textit{ iff } h \textit{ is real}, \\
& \quad e^{itA} \textit{ is unitary for all real } t.
\end{aligned}
$$

Spectral Theorem (continued)

Theorem

$$\begin{aligned} & \chi_\Gamma(A) \textit{ is an orthogonal projection for each Borel set } \Gamma, \\ E(\lambda) &= \chi_{(-\infty,\lambda]}(A), \quad \lambda \in \mathbb{R}, \quad \textit{resolution of the identity} \\ A &= \int_{\mathbb{R}} \lambda dE(\lambda), \qquad h(A) = \int_{\mathbb{R}} h(\lambda) dE(\lambda). \end{aligned}$$

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- Get $\langle L(A)f, g\rangle$ from $\langle L(A)h, h\rangle$,

$$\langle L(A)h, h\rangle = \int_{\mathbb{R}} L(\lambda) d\, \|E(\lambda)h\|^2 .$$

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$$\begin{aligned} K(t) &= \left\| u'(t) \right\|^2, \\ P(t) &= \| Au(t) \|^2 = \left\langle A^2 u, u \right\rangle, \\ E &= K + P. \end{aligned}$$

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- Use

$$\frac{d}{dt} \langle w, w \rangle = 2\,\mathrm{Re} \left\langle w', w \right\rangle$$

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Energy conservation

- $$\begin{aligned} \frac{dE}{dt} &= 2\,\mathrm{Re}\{\langle u'', u'\rangle + \langle Au, Au'\rangle\} \\ &= 2\,\mathrm{Re}\,\langle u'' + A^2 u, u'\rangle = 0. \end{aligned}$$

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- $$\begin{aligned} K(t) &= \left\| i(e^{itA}AF - e^{-itA}AG) \right\|^2 \\ P(t) &= \left\| e^{itA}AF + e^{-itA}AG \right\|^2 \end{aligned}$$

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- $$K - P = -4\,\mathrm{Re}\left\langle e^{itA}AF, e^{-itA}AG \right\rangle = -4\,\mathrm{Re}\left\langle e^{2itA}H, K \right\rangle.$$

Equipartition of energy

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- Hence for all finite energy solutions, equipartition of energy holds

$$\begin{aligned} K - P &\rightarrow 0 \text{ as } t \rightarrow \infty \\ \text{iff } \left\langle e^{itA} h, h \right\rangle &\rightarrow 0 \text{ as } t \rightarrow \infty \text{ for all } h. \end{aligned}$$

Equipartition of energy II

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  - ν_d is discrete iff it is atomic and its "distribution function" is pure jump.

Lebesgue-Kato decomposition

- For A selfadjoint on H,

$$H_j(A) = \{h \in H : d\,\|E(\lambda)h\|^2 \text{ is } j\} \text{ for } j = ac, sc, d,$$

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- Equipartition of energy holds iff $H_{RL}(A) = H$ (JG, 1969)

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- $E(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Equipartition of energy???

Theorem

(JG and Jim Sandefur, 1987) $0 < a < b$, $H_{ac}(A) = H$ *implies*

$$\frac{K(t)}{P(t)} \to 1 \text{ as } t \to \infty \text{ for all nonzero finite energy solutions.}$$

Strong damping

- New research started in 2009

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$$B = aA^{\alpha},\ \ a > 0,\ \ 0 \leq \alpha < 1$$

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- Note that for $B = aI$ (telegraph case)

$$\frac{\widehat{K}(t)}{\widehat{P}(t)} = \frac{K(t)}{P(t)}.$$

Classical Overdamping

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- For such solutions y,
  $$\begin{aligned} |y(t)| &\leq Ce^{-at} \text{ if } a < \omega, \\ |y(t)| &\leq Ce^{-(a-\sqrt{a^2-\omega^2})t} \text{ if } a > \omega. \end{aligned}$$

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- Here $C = C(y(0), y'(0))$. We omit the double root case $a = \omega$.

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Overdamping II

- GG-JG-Perla

$$u'' + 2au' + A^2 u = 0,$$

$$A = A^* \geq bI,$$

$$b = \inf \sigma(A) > 0, a > 0.$$

Theorem

(GGP) *Optimal estimate:*

$$E(t) \leq Ce^{-\gamma t}$$

$$\gamma = a \text{ if } a < b$$

$$\gamma = a - \sqrt{a^2 - \omega^2} \text{ if } a > b.$$

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Let $B = aA^{\alpha}$, $0 \leq \alpha < 1$. This damped wave equation is a ψDE. To make it a PDE, take

$$A^2 = -D^6 + \epsilon^2 I, \quad D = d/dx, \quad H = L^2(\mathbb{R}), \; B = (A^2 - \epsilon^2 I)^{2/3} = -D^2$$

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- $C = C_0(E(0))$ holds on some subspaces but fails in general.

Overdamping theorem

Theorem

(GG, JG & GR) $B = F(A)$, $0 < b = \inf \sigma(A)$, *F is continuous on* $(0{,}\infty)$; *F is bounded on (0,1), and there is a* $\gamma > 0$ *such that*

$$\begin{aligned} F(x) &> x \text{ on } (0,\gamma) \\ F(x) &< x \text{ on } (\gamma,\infty) \\ (1-\epsilon)x - F(x) &> 0 \text{ for large } x. \end{aligned}$$

Then

$$E(t) \leq Ce^{-\alpha(\gamma)t} \text{ holds with}$$

$$\alpha(\gamma) = \begin{cases} \text{essinf } f_{[b,\infty)}(F) & \text{if} \quad \gamma < b \\ \min\{\text{essinf } f_{[b,\gamma)}(C_1), \text{essinf } f_{[\gamma,\infty)}(F)\} & \text{if} \quad \gamma > b \end{cases}$$

$$C_1 = B - [B^2 - A^2]^{1/2} = C_1(A)$$

Theorem (continued)

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For $B = aA^\alpha$ as before,

$$\alpha(\gamma) = \begin{cases} ab^\alpha & \text{if} \quad \gamma = a^{\frac{1}{1-\alpha}} < b \\ ab^\alpha - b^\alpha\sqrt{a^2 - b^{2-2\alpha}} & \text{if} \quad \gamma = a^{\frac{1}{1-\alpha}} > b \end{cases}.$$

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  - Raluca Clendenen (2013 PhD)

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$$u(t) = v(t)\{1 + o(t)\} \text{ as } t \to \infty \quad \text{provided } h \neq 0.$$

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Dynamic boundary conditions

- Consider

$$u_t = \Delta u \ \text{ in } \Omega,$$

$$u_t + \beta \frac{\partial u}{\partial n} + \gamma u - q\beta \Delta_{LB} u = 0 \text{ on } \partial\Omega$$

- Diffusion inside & diffusion on the boundary
- If $u_t = \Delta u$ holds on $\bar{\Omega} \times \mathbb{R}^+$, we can replace u_t by Δu in the boundary condition.

Wentzell boundary conditions

Example

Let Ω be an unbounded domain in $\mathbb{R}^N$ containing arbitrarily large balls. Consider the "Wentzell Laplacian"

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- The nature of $\sigma(A)$ is unknown.

Linear unidirectional waves in 1 D

- Consider, for $x \in \mathbb{R}$

$$u_t = c_1 u_x + c_2 u_{xxx}$$
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